

Off-shell formulation of $N = 2$ Super Yang–Mills theories coupled to matter without auxiliary fields

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ABSTRACT

$N=2$ supersymmetric Yang–Mills theories coupled to matter are considered in the Wess–Zumino gauge. The supersymmetries are realized nonlinearly and the anticommutator between two susy charges gives, in addition to translations, gauge transformations and equations of motion. The difficulties hidden in such an algebraic structure are well known: almost always auxiliary fields can be introduced in order to put the formalism off-shell, but still the field dependent gauge transformations give rise to an infinite dimensional algebra quite hard to deal with. However, it is possible to avoid all these problems by collecting into a unique nilpotent operator all the symmetries defining the theory, namely ordinary BRS, supersymmetries and translations. According to this method the role of the auxiliary fields is covered by the external sources coupled, as usual, to the nonlinear variations of the quantum fields. The analysis is then formally reduced to that of ordinary Yang–Mills theory.

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1 Introduction

Amongst the Supersymmetric Yang – Mills (SYM) theories, the extended $N = 2$ models [1] play a particular role, because of the number of exact results that can be extracted from them. For instance, a new mechanism for chiral symmetry breaking deriving from the condensation of magnetic monopoles has been investigated for $N = 2$ SYM in [2], where also an analysis of the electric magnetic duality is done. Another reason of interest is their generality: in fact, the reduction to the $N = 1$ case being straightforward, also the maximally extended $N = 4$ SYM can be considered as a $N = 2$ theory, with matter in the adjoint representation of the gauge group.

In the Wess–Zumino gauge, the supersymmetry transformations are nonlinear, and consequently the major problem affecting the model is related to its algebraic structure – described in Section 2 –, which involves equations of motion and field dependent gauge transformations. This gives rise to an infinite dimensional algebra, even if auxiliary fields can be introduced to put the formalism off-shell [3].

Another difficulty consists in the definition of a gauge fixing term, which, in the usual framework, is BRS invariant by construction as it is BRS exact, but on the other hand it cannot be supersymmetric.

These two problems of SYMs, namely the existence of an infinite dimensional algebraic structure and the non invariance of the gauge fixing term, turn out to be somehow related

to each other, because they can both be solved at the same time.

In Section 3 we show indeed that it is possible to define a generalized BRS operator which sums up the usual BRS invariance, the supersymmetry and the invariance of the theory under translations. The method we are going to exploit is quite general and it is particularly powerful in view of the renormalization of models exhibiting non trivial algebraic structures, not only of the supersymmetric type. In that direction, it has been already successfully applied to topological [4], supersymmetric [5, 6], ordinary [7] gauge field theories as well as to non gauge field theories [8]. In particular in [6], this technique has been used to study the renormalization of the $N = 1$ and $N = 4$ case. Here we generalize the description to the $N = 2$ model, as a preliminary and necessary step for the discussion of its quantum extension [9].

The essence consists simply in collecting all the symmetries of the theory into an unique operator. Opportune transformations of the ghost fields make this generalized BRS operator on-shell nilpotent, allowing us to construct an invariant gauge-fixing term. This permits us to achieve the main purpose of this paper, *i.e.* to completely determine the $N = 2$ SYMs at the classical level, by means of a classical gauge fixed action invariant under a Slavnov-like operator which summarizes all the symmetries of the theory.

In Section 4 we solve classically the constraints which define the theory. At the same time we give the expressions of the operators which will be used in the algebraic renormalization of this model to constrain the counterterms and the possible anomalies [9].

Results and perspectives are briefly discussed in the concluding Section 5.

2 The model

The massless irreducible representations of $N = 2$ supersymmetry in four dimensions are realized by means of the Super Yang – Mills (SYM) multiplet and the matter multiplet

$$\begin{array}{lllll} \text{SYM} & A_\mu^a & \lambda_{\alpha i}^a & A^a & B^a \\ \text{matter} & A^{iA} & A_{iA}^* & \psi_A^\alpha & \bar{\psi}_\alpha^A \end{array}$$

Here A_μ^a is the gauge field and $\lambda_{\alpha i}^a$ is a doublet of Majorana spinors, for which the following Majorana condition holds

$$\lambda_{\alpha i}^a = (i\gamma_5 C)_{\alpha\beta} \epsilon_{ij} \bar{\lambda}^{\beta j} , \quad (2.1)$$

where C is the charge conjugation matrix ($C\gamma^\mu C^{-1} = -\gamma^{\mu T}$) and ϵ_{ij} is the two-dimensional Levi–Civita tensor. The spinless fields A^a and B^a are a scalar and a pseudoscalar respectively; the matter fields consist in a doublet of complex scalar fields (A^{iA}, A_{iA}^*) and two Dirac spinors ($\psi_A^\alpha, \bar{\psi}_\alpha^A$).

The index μ describes minkowskian spacetime, α is the spinorial index, which runs from 1 to 4, and $i = 1, 2$ is the supersymmetry index. The $N = 2$ SYM multiplet belongs

to the adjoint representation of a gauge group G , while the matter multiplet can be in any representation, which we assume to be real. The indices a and A run over the corresponding Lie algebras. As it is well known, when also the matter multiplet belongs to the adjoint representation of the gauge group, $N = 4$ supersymmetry is recovered.

In the Wess – Zumino gauge, the supersymmetry is realized nonlinearly [10]

$$\begin{aligned}
\delta A^a &= \bar{\varepsilon}^i \lambda_i^a \\
\delta B^a &= i \bar{\varepsilon}^i \gamma_5 \lambda_i^a \\
\delta A_\mu^a &= \bar{\varepsilon}^i \gamma_\mu \lambda_i^a \\
\delta \lambda_{\alpha i}^a &= \frac{1}{2} F_{\mu\nu}^a (\sigma^{\mu\nu} \varepsilon_i)_\alpha - (D_\mu A)^a (\gamma^\mu \varepsilon_i)_\alpha \\
&\quad + i (D_\mu B)^a (\gamma^\mu \gamma_5 \varepsilon_i)_\alpha + i f^{abc} A^b B^c (\gamma_5 \varepsilon_i)_\alpha
\end{aligned} \tag{2.2}$$

$$\begin{aligned}
\delta A^{iA} &= \bar{\varepsilon}^i \psi^A \\
\delta \psi_\alpha^A &= -2 (D_\mu A^i)^A (\gamma^\mu \varepsilon_i)_\alpha + 2 (T^a)_B^A A^{iB} A^a \varepsilon_{\alpha i} \\
&\quad - 2i (T^a)_B^A A^{iB} B^a (\gamma_5 \varepsilon_i)_\alpha ,
\end{aligned} \tag{2.3}$$

where $\sigma_{\mu\nu} \equiv \frac{1}{2} [\gamma_\mu, \gamma_\nu]$, and $(T^a)_B^A$ are the generators of the gauge group G .

As usual, the nonlinearities are in the spinor transformation laws, which depend on the field strenght

$$F_{\mu\nu}^a \equiv \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + f^{abc} A_\mu^b A_\nu^c \tag{2.4}$$

and on the covariant derivatives

$$\begin{aligned}(D_\mu X)^a &\equiv \partial_\mu X^a + f^{abc} A_\mu^b X^c \\ (D_\mu Y)^A &\equiv \partial_\mu Y^A + (T^a)_B^A A_\mu^a Y^B .\end{aligned}\tag{2.5}$$

The relation between the supersymmetry charges $Q_{\alpha i}$ and the above supersymmetry operator δ is

$$\delta \equiv \bar{\varepsilon}^{\alpha i} Q_{\alpha i} ,\tag{2.6}$$

namely $\varepsilon_{\alpha i}$ is an infinitesimal fermionic anticommuting Majorana parameter

$$\varepsilon_{\alpha i} = (i\gamma_5 C)_{\alpha\beta} \epsilon_{ij} \bar{\varepsilon}^{\beta j} .\tag{2.7}$$

The action

$$S = S_{SYM} + S_{matter} + S_{int} ,\tag{2.8}$$

where

$$\begin{aligned}S_{SYM} = \frac{1}{g^2} \int d^4x \Bigg(& -\frac{1}{4} F^{a\mu\nu} F_{\mu\nu}^a + \frac{1}{2} (D^\mu A)^a (D_\mu A)^a + \frac{1}{2} (D^\mu B)^a (D_\mu B)^a \\ & + \frac{1}{2} \bar{\lambda}^{ai} \gamma^\mu (D_\mu \lambda_i)^a + \frac{1}{2} f^{abc} (\bar{\lambda}^{bi} \lambda_i^c) A^a \\ & - i \frac{1}{2} f^{abc} (\bar{\lambda}^{bi} \gamma_5 \lambda_i^c) B^a - \frac{1}{2} f^{amn} A^m B^n f^{apq} A^p B^q \Bigg) ,\end{aligned}\tag{2.9}$$

$$S_{matter} = \int d^4x \left((D^\mu A^i)^A (D_\mu A_i^*)_A - \frac{1}{2} \bar{\psi}_A \gamma^\mu (D_\mu \psi)^A \right) ,\tag{2.10}$$

$$\begin{aligned}
S_{int} = \int d^4x \Bigg(& -(T^a)_B^A (\bar{\psi}_A \lambda_i^a) A^{iB} + (T^a)_B^A (\bar{\lambda}^{ai} \psi^B) A_{iA}^* \\
& + (T^a)_D^A (T^b)_B^D A^a A^b A_{iA}^* A^{iB} + (T^a)_D^A (T^b)_B^D B^a B^b A_{iA}^* A^{iB} \\
& - i \frac{1}{2} (T^a)_B^A (\bar{\psi}_A \gamma_5 \psi^B) B^a - \frac{1}{2} (T^a)_B^A (\bar{\psi}_A \psi^B) A^a \Bigg) ,
\end{aligned} \tag{2.11}$$

and g^2 is the only coupling constant of the theory, is invariant under supersymmetry

$$\delta S = 0 . \tag{2.12}$$

The algebra formed by the supersymmetry transformations (2.2) and (2.3) exhibits two obstructions to its closure on the translations :

$$\begin{aligned}
[\delta_1, \delta_2] \Phi &= 2(\bar{\varepsilon}_1^i \gamma^\mu \varepsilon_{2i}) \partial_\mu \Phi \\
&+ 2\delta_{\text{gauge}}(\omega^a) \Phi \\
&+ \text{field equations} ,
\end{aligned} \tag{2.13}$$

where Φ stands for all the fields of the theory.

The second term on the r.h.s. of equation (2.13) is a gauge transformation with a *field dependent* parameter ω^a

$$\omega^a \equiv (\bar{\varepsilon}_1^i \gamma^\mu \varepsilon_{2i}) A_\mu + i(\bar{\varepsilon}_1^i \gamma_5 \varepsilon_{2i}) B - (\bar{\varepsilon}_1^i \varepsilon_{2i}) A , \tag{2.14}$$

and the last term on the r.h.s of (2.13) is a contact term present only when Φ is one of the spinors of the theory, namely λ or ψ .

3 The strategy

The algebraic structure described by (2.13) is typical of nonlinearly realized supersymmetries and the general attitude when dealing with it is to put the formalism off-shell firstly, by adding a suitable number of auxiliary fields.

This causes a few drawbacks, the first of which, as it is apparent from (2.13), is related to the fact that still one is left with an algebra closing on translations modulo the field dependent gauge transformations (2.14). This algebra is infinite-dimensional and requires an infinite number of composite operators of increasing negative dimensions in order to be controlled. As a consequence the discussion of the renormalization of the model, which is our ultimate aim [9], becomes difficult, if not impossible [3]. In addition to that, no gauge fixing term can be found which is invariant under supersymmetry and therefore one has also to deal with the consequent breaking. Finally, not always auxiliary fields can be introduced to eliminate the field equations from the algebra (2.13). In fact, if the matter multiplet belongs to the adjoint representation of the gauge group, the theory can be interpreted as having a $N = 4$ supersymmetry, for which no off-shell formulation through auxiliary fields is known [10].

These are the reasons why we prefer to follow here an approach which does not rely on the existence of auxiliary fields. In order to do that, we first concentrate on the elimination from the algebra of the field dependent gauge transformations (2.14), temporarily disregarding the presence of the equations of motion in (2.13). We shall get rid of them subsequently, without introducing auxiliary fields.

Besides being supersymmetric, the action S (2.8) is left invariant under the usual BRS transformations

$$\begin{aligned}
sA^a &= f^{abc}c^bA^c \\
sB^a &= f^{abc}c^bB^c \\
sA_\mu^a &= -(D_\mu c)^a \\
s\lambda_{\alpha i}^a &= f^{abc}c^b\lambda_{\alpha i}^c \\
sA^{iA} &= (T^a)_B^A c^a A^{iB} \\
s\psi_\alpha^A &= (T^a)_B^A c^a \psi_\alpha^B \\
sc^a &= \frac{1}{2}f^{abc}c^b c^c \\
s\bar{c}^a &= b^a \quad sb^a = 0 \ ,
\end{aligned} \tag{3.1}$$

where c , \bar{c} and b are respectively the ghost, the antighost and the Lagrange multiplier, which is introduced to implement the gauge condition

$$\partial^\mu A_\mu^a = 0 \ . \tag{3.2}$$

The BRS transformations (3.1) are nilpotent

$$s^2 = 0 , \quad (3.3)$$

and they commute with the supersymmetry transformations (2.2)-(2.3), trivially extended to c , \bar{c} and b

$$[s, \delta] = 0 . \quad (3.4)$$

The canonical dimensions and the Faddeev–Popov charges of the quantum fields are reported in Table 1

	A_μ^a	$\lambda_{\alpha i}^a$	A^a	B^a	A^{iA}	ψ_A^α	c^a	\bar{c}^a	b^a
dim	1	3/2	1	1	1	3/2	0	2	2
$\Phi\Pi$	0	0	0	0	0	0	1	-1	0

Table 1. Dimensions and Faddeev–Popov charges of the quantum fields.

The BRS and supersymmetry operators form the algebra represented by the relations (2.13), (3.3) and (3.4). Now we collect s and $\delta \equiv \bar{\varepsilon}^i Q_i$ together, defining a new operator \mathcal{Q}

$$\mathcal{Q} \equiv s + \bar{\varepsilon}^i Q_i + \xi^\mu \partial_\mu - (\bar{\varepsilon}^i \gamma^\mu \varepsilon_i) \frac{\partial}{\partial \xi^\mu} , \quad (3.5)$$

where we introduced two global ghosts ε_i and ξ^μ , associated to supersymmetry and translations respectively. In order to have an homogeneous operator \mathcal{Q} , we must assign the

following dimensions and Faddeev–Popov charges²

	ε_i	ξ^μ
dim	$-1/2$	-1
$\Phi\Pi$	1	1

Table 2. Dimensions and Faddeev–Popov charges of the global ghosts.

The quantum fields transform under \mathcal{Q} as follows

$$\begin{aligned}
\mathcal{Q}A^a &= f^{abc}c^bA^c + \bar{\varepsilon}^i\lambda_i^a + \xi^\mu\partial_\mu A^a \\
\mathcal{Q}B^a &= f^{abc}c^bB^c + i\bar{\varepsilon}^i\gamma_5\lambda_i^a + \xi^\mu\partial_\mu B^a \\
\mathcal{Q}A_\mu^a &= -(D_\mu c)^a + \bar{\varepsilon}^i\gamma_\mu\lambda_i^a + \xi^\nu\partial_\nu A_\mu^a \\
\mathcal{Q}\lambda_{\alpha i}^a &= f^{abc}c^b\lambda_{\alpha i}^c + \frac{1}{2}F_{\mu\nu}^a(\sigma^{\mu\nu}\varepsilon_i)_\alpha - (D_\mu A)^a(\gamma^\mu\varepsilon_i)_\alpha \\
&\quad + i(D_\mu B)^a(\gamma^\mu\gamma_5\varepsilon_i)_\alpha + if^{abc}A^bB^c(\gamma_5\varepsilon_i)_\alpha + \xi^\mu\partial_\mu\lambda_{\alpha i}^a \\
\mathcal{Q}A^{iA} &= (T^a)_B^A c^a A^{iB} + \bar{\varepsilon}^i\psi^A + \xi^\mu\partial_\mu A^{iA} \\
\mathcal{Q}\psi_\alpha^A &= (T^a)_B^A c^a \psi_\alpha^B - 2(D_\mu A^i)^A(\gamma^\mu\varepsilon_i)_\alpha + 2(T^a)_B^A A^{iB}A^a\varepsilon_{\alpha i} \\
&\quad - 2i(T^a)_B^A A^{iB}B^a(\gamma_5\varepsilon_i)_\alpha + \xi^\mu\partial_\mu\psi_\alpha^A \\
\mathcal{Q}c^a &= \frac{1}{2}f^{abc}c^b c^c - (\bar{\varepsilon}^i\gamma^\mu\varepsilon_i)A_\mu^a - i(\bar{\varepsilon}^i\gamma_5\varepsilon_i)B^a + (\bar{\varepsilon}^i\varepsilon_i)A^a + \xi^\mu\partial_\mu c^a \\
\mathcal{Q}\bar{c}^a &= b^a + \xi^\mu\partial_\mu\bar{c}^a
\end{aligned} \tag{3.6}$$

²The following commutation rule holds for elements ϕ_i with ghost charge g_i and spinorial Grassmann parity p_i : $\phi_1\phi_2 = (-1)^{g_1g_2+p_1p_2}\phi_2\phi_1$.

$$\begin{aligned}
\mathcal{Q}b^a &= (\bar{\varepsilon}^i \gamma^\mu \varepsilon_i) \partial_\mu \bar{c}^a + \xi^\mu \partial_\mu b^a \\
\mathcal{Q}\xi^\mu &= -(\bar{\varepsilon}^i \gamma^\mu \varepsilon_i) \\
\mathcal{Q}\varepsilon_i &= 0 .
\end{aligned}$$

The action of \mathcal{Q} on the fields belonging to the SYM and matter multiplet follows trivially from their BRS (3.1) and supersymmetry (2.2)-(2.3) transformations. Notice that we let the ghost field c transform into the field dependent gauge transformation (2.14), in addition to its usual BRS variation and translation. In this way, we reach our first goal, namely the elimination from the algebra of the field dependent gauge transformations. The result is that the operator \mathcal{Q} is on-shell nilpotent

$$\mathcal{Q}^2 = \text{equations of motion} . \quad (3.7)$$

Explicitely we get

$$\begin{aligned}
\mathcal{Q}^2 \Phi &= 0 \quad \Phi = \text{all fields but } \lambda, \psi \\
\mathcal{Q}^2 \lambda_{\alpha i}^a &= 2 \left(\bar{\varepsilon}^j \frac{\delta S_{SYM}}{\delta \bar{\lambda}^{aj}} \right) \varepsilon_{\alpha j} - \left(\bar{\varepsilon}^j \frac{\delta S_{SYM}}{\delta \bar{\lambda}^{aj}} \right) \varepsilon_{\alpha i} \\
\mathcal{Q}^2 \psi_\alpha^A &= -(\bar{\varepsilon}^i \varepsilon_i) \frac{\delta S}{\delta \bar{\psi}_A^\alpha} + (\bar{\varepsilon}^i \gamma_5 \varepsilon_i) \left(\gamma_5 \frac{\delta S}{\delta \bar{\psi}_A} \right)_\alpha + (\bar{\varepsilon}^i \gamma^\mu \varepsilon_i) \left(\gamma_\mu \frac{\delta S}{\delta \bar{\psi}_A} \right)_\alpha .
\end{aligned} \quad (3.8)$$

The operator \mathcal{Q} obviously describes a symmetry of the action S

$$\mathcal{Q}S = 0 . \quad (3.9)$$

By means of the operator \mathcal{Q} , which is nilpotent on all fields but the spinors, we can construct an invariant gauge fixing term in the usual way, namely

$$\begin{aligned} S_{gf} &= \mathcal{Q} \int d^4x \bar{c}^a \left(\partial A^a + \frac{\theta}{2} b^a \right) \\ &= \int d^4x \left(b^a \partial A^a + \frac{\theta}{2} b^2 - (\partial^\mu \bar{c}^a) (D_\mu c)^a + (\partial^\mu \bar{c}^a) (\bar{\varepsilon}^i \gamma_\mu \lambda_i^a) \right), \end{aligned} \quad (3.11)$$

where θ is the gauge parameter, which is zero in the Landau gauge.

We thus end up with a symmetry of the action

$$S_{inv} \equiv S + S_{gf}, \quad (3.11)$$

described by an operator, \mathcal{Q} , which is on-shell nilpotent. This situation is sometimes called of Batalin – Vilkovisky type. We can bypass the general procedure [11] devised to quantize these kind of models: it is in fact well known [12, 13] that, in order to write the Slavnov identity corresponding to an on-shell nilpotent symmetry, one must add to the action terms of higher order in the external sources which are coupled, as usual, to the nonlinear variations of the quantum fields.

In our case the source dependent term of the action must be

$$\begin{aligned} S_{ext} = \int d^4x \left(M^a(\mathcal{Q}A^a) + N^a(\mathcal{Q}B^a) + \Omega^{a\mu}(\mathcal{Q}A_\mu^a) + \bar{\Lambda}^{ai}(\mathcal{Q}\lambda_i^a) + L^a(\mathcal{Q}c^a) \right. \\ \left. + U_{iA}^*(\mathcal{Q}A^{iA}) + U^{iA}(\mathcal{Q}A_{iA}^*) + \bar{\Psi}_A(\mathcal{Q}\psi^A) + (\mathcal{Q}\bar{\psi}_A)\Psi^A \right. \\ \left. - (\bar{\varepsilon}^j \Lambda_i^a)(\bar{\Lambda}^{ai} \varepsilon_j) + \frac{1}{2}(\bar{\varepsilon}^j \Lambda_j^a)(\bar{\Lambda}^{ai} \varepsilon_i) \right) \end{aligned} \quad (3.12)$$

$$-(\bar{\varepsilon}^i \varepsilon_i)(\bar{\Psi}_A \Psi^A) + (\bar{\varepsilon}^i \gamma_5 \varepsilon_i)(\bar{\Psi}_A \gamma_5 \Psi^A) + (\bar{\varepsilon}^i \gamma^\mu \varepsilon_i)(\bar{\Psi}_A \gamma_\mu \Psi^A) \Big) ,$$

notice the nonstandard quadratic term in the external sources Λ and Ψ .

The total classical action

$$\Sigma \equiv S_{SYM} + S_{matter} + S_{int} + S_{gf} + S_{ext} \quad (3.13)$$

satisfies the Slavnov identity

$$\mathcal{S}(\Sigma) = 0 , \quad (3.14)$$

where

$$\begin{aligned} \mathcal{S}(\Sigma) = \int d^4x \Big(& \frac{\delta \Sigma}{\delta \Omega^{a\mu}} \frac{\delta \Sigma}{\delta A_\mu^a} + \frac{\delta \Sigma}{\delta L^a} \frac{\delta \Sigma}{\delta c^a} + \frac{\delta \Sigma}{\delta M^a} \frac{\delta \Sigma}{\delta A^a} + \frac{\delta \Sigma}{\delta N^a} \frac{\delta \Sigma}{\delta B^a} + \frac{\delta \Sigma}{\delta \bar{\Lambda}^{ai}} \frac{\delta \Sigma}{\delta \lambda_i^a} \\ & + \frac{\delta \Sigma}{\delta U_{iA}^*} \frac{\delta \Sigma}{\delta A^{iA}} + \frac{\delta \Sigma}{\delta U^{iA}} \frac{\delta \Sigma}{\delta A_{iA}^*} + \frac{\delta \Sigma}{\delta \bar{\Psi}_A} \frac{\delta \Sigma}{\delta \psi^A} + \frac{\delta \Sigma}{\delta \Psi^A} \frac{\delta \Sigma}{\delta \bar{\psi}_A} \\ & + (b^a + \xi^\mu \partial_\mu \bar{c}^a) \frac{\delta \Sigma}{\delta \bar{c}^a} + ((\bar{\varepsilon}^i \gamma^\mu \varepsilon_i) \partial_\mu \bar{c}^a + \xi^\mu \partial_\mu b^a) \frac{\delta \Sigma}{\delta b^a} \Big) \\ & - (\bar{\varepsilon}^i \gamma^\mu \varepsilon_i) \frac{\partial \Sigma}{\partial \xi^\mu} . \end{aligned} \quad (3.15)$$

The corresponding linearized Slavnov operator

$$\begin{aligned} \mathcal{B}_\Sigma = \int d^4x \Big(& \frac{\delta \Sigma}{\delta \Omega^{a\mu}} \frac{\delta}{\delta A_\mu^a} + \frac{\delta \Sigma}{\delta A_\mu^a} \frac{\delta}{\delta \Omega^{a\mu}} + \frac{\delta \Sigma}{\delta L^a} \frac{\delta}{\delta c^a} + \frac{\delta \Sigma}{\delta c^a} \frac{\delta}{\delta L^a} + \frac{\delta \Sigma}{\delta M^a} \frac{\delta}{\delta A^a} + \frac{\delta \Sigma}{\delta A^a} \frac{\delta}{\delta M^a} \\ & + \frac{\delta \Sigma}{\delta N^a} \frac{\delta}{\delta B^a} + \frac{\delta \Sigma}{\delta B^a} \frac{\delta}{\delta N^a} + \frac{\delta \Sigma}{\delta \bar{\Lambda}^{ai}} \frac{\delta}{\delta \lambda_i^a} - \frac{\delta \Sigma}{\delta \lambda_i^a} \frac{\delta}{\delta \bar{\Lambda}^{ai}} + \frac{\delta \Sigma}{\delta U_{iA}^*} \frac{\delta}{\delta A^{iA}} + \frac{\delta \Sigma}{\delta A^{iA}} \frac{\delta}{\delta U_{iA}^*} \\ & + \frac{\delta \Sigma}{\delta U^{iA}} \frac{\delta}{\delta A_{iA}^*} + \frac{\delta \Sigma}{\delta A_{iA}^*} \frac{\delta}{\delta U^{iA}} + \frac{\delta \Sigma}{\delta \bar{\Psi}_A} \frac{\delta}{\delta \psi^A} - \frac{\delta \Sigma}{\delta \psi^A} \frac{\delta}{\delta \bar{\Psi}_A} + \frac{\delta \Sigma}{\delta \Psi^A} \frac{\delta}{\delta \bar{\psi}_A} - \frac{\delta \Sigma}{\delta \bar{\psi}_A} \frac{\delta}{\delta \Psi^A} \end{aligned}$$

$$\begin{aligned}
& + (b^a + \xi^\mu \partial_\mu \bar{c}^a) \frac{\delta}{\delta \bar{c}^a} + ((\bar{\varepsilon}^i \gamma^\mu \varepsilon_i) \partial_\mu \bar{c}^a + \xi^\mu \partial_\mu b^a) \frac{\delta}{\delta b^a} \Big) \\
& - (\bar{\varepsilon}^i \gamma^\mu \varepsilon_i) \frac{\partial}{\partial \xi^\mu}
\end{aligned} \tag{3.16}$$

is off-shell nilpotent

$$\mathcal{B}_\Sigma \mathcal{B}_\Sigma = 0 . \tag{3.17}$$

The form of the linearized Slavnov operator (3.16) implies that the action of \mathcal{B}_Σ on the quantum fields is given by

$$\mathcal{B}_\Sigma \Phi = \frac{\delta \Sigma}{\delta K_\Phi} , \tag{3.18}$$

where K_Φ are the external sources coupled to the non linear \mathcal{Q} -variations of the quantum fields Φ . Therefore $\mathcal{B}_\Sigma \Phi$ coincides with $\mathcal{Q}\Phi$ for all the fields but the spinors, for which we have

$$\begin{aligned}
\mathcal{B}_\Sigma \lambda_{\alpha i}^a &= \mathcal{Q} \lambda_{\alpha i}^a - 2(\bar{\varepsilon}^i \Lambda_i^a) \varepsilon_{\alpha j} + (\bar{\varepsilon}^j \Lambda_j^a) \varepsilon_{\alpha i} \\
\mathcal{B}_\Sigma \psi_\alpha^A &= \mathcal{Q} \psi_\alpha^A - (\bar{\varepsilon}^i \varepsilon_i) \Psi_\alpha^A + (\bar{\varepsilon}^i \gamma_5 \varepsilon_i) (\gamma_5 \Psi^A)_\alpha + (\bar{\varepsilon}^i \gamma^\mu \varepsilon_i) (\gamma_\mu \Psi^A)_\alpha ,
\end{aligned} \tag{3.19}$$

and

$$\mathcal{B}_\Sigma^2 \lambda_{\alpha i}^a = \mathcal{B}_\Sigma^2 \psi_\alpha^A = 0 . \tag{3.20}$$

Comparing (3.20) with (3.8), it is apparent that the effect of adding a bilinear term in the external sources to the action is to modify the transformations laws of the spinor fields in order to obtain the nilpotency. In this sense, the external sources, whose presence is in any case necessary in view of the quantization of the theory, in addition play the same

role of the auxiliary fields usually introduced to put the formalism off-shell.

4 The semiclassical approximation

The Slavnov identity (3.14) alone is not sufficient to uniquely determine the classical action. The theory is defined also by the following constraints :

1. the gauge condition

$$\frac{\delta \Sigma}{\delta b^a} = \partial A^a + \theta b^a ; \quad (4.1)$$

in the Landau gauge, the commutator between the gauge condition (4.1) and the Slavnov identity (3.14) gives the antighost equation

$$\frac{\delta \Sigma}{\delta \bar{c}^a} + \partial^\mu \frac{\delta \Sigma}{\delta \Omega^{a\mu}} - \xi^\mu \partial_\mu \frac{\delta \Sigma}{\delta b^a} = 0 ; \quad (4.2)$$

2. the ξ -equation

$$\frac{\partial \Sigma}{\partial \xi^\mu} = \Delta_\mu ; \quad (4.3)$$

where

$$\begin{aligned} \Delta_\mu \equiv \int d^4x \bigg(& -M^a \partial_\mu A^a - N^a \partial_\mu B^a - \Omega^{a\nu} \partial_\mu A_\nu^a - (\bar{\Lambda}^{ai} \partial_\mu \lambda_i^a) + L^a \partial_\mu c^a \\ & - U_{iA}^* \partial_\mu A^{iA} - U^{iA} \partial_\mu A_{iA}^* - (\bar{\Psi}_A \partial_\mu \psi^A) + (\partial_\mu \bar{\psi}_A \Psi^A) \bigg) . \end{aligned} \quad (4.4)$$

By anticommuting the ξ -equation with the Slavnov identity (3.14), one gets the Ward identity for the translations

$$\frac{\partial}{\partial \xi^m} \mathcal{S}(\Sigma) + \mathcal{B}_\Sigma \left(\frac{\partial \Sigma}{\partial \xi^\mu} - \Delta_\mu \right) = \sum_{all \ fields} \int d^4x (\partial_\mu \Phi) \frac{\delta \Sigma}{\delta \Phi} \equiv \mathcal{P}_\mu \Sigma = 0 ; \quad (4.5)$$

3. the ghost equation, peculiar to the Landau gauge $\theta = 0$ [14, 13]

$$\mathcal{F}^a \Sigma = \Delta^a , \quad (4.6)$$

where

$$\mathcal{F}^a \equiv \int d^4x \left(\frac{\delta}{\delta c^a} + f^{abc} \bar{c}^b \frac{\delta}{\delta b^c} \right) \quad (4.7)$$

and

$$\begin{aligned} \Delta^a \equiv \int d^4x \left(f^{abc} \left(M^b A^c + N^b B^c + \Omega^{b\mu} A_\mu^c + (\bar{\Lambda}^{bi} \lambda_i^c) - L^b c^c \right) \right. \\ \left. - (T^a)_B^A \left(U_{iA}^* A^{iB} - U^{iB} A_{iA}^* + (\bar{\Psi}_A \psi^B) + (\bar{\psi}_A \Psi^B) \right) \right) . \end{aligned} \quad (4.8)$$

The ghost equation (4.6), anticommutated with the Slavnov identity (3.14), gives the Ward identity of the rigid gauge invariance

$$\mathcal{F}^a \mathcal{S}(\Sigma) + \mathcal{B}_\Sigma (\mathcal{F}^a \Sigma - \Delta^a) = \sum_{all \ fields} \int d^4x (\delta_{rig} \Phi) \frac{\delta \Sigma}{\delta \Phi} \equiv \mathcal{H}_{rig}^a \Sigma = 0 , \quad (4.9)$$

where $\delta_{rig} \Phi$ are the rigid gauge transformations of the fields Φ , described by the BRS transformations (3.1) with c constant parameter.

The constraints (4.1), (4.3) and (4.6) have the form of broken symmetries of the action Σ , but notice that the breakings are linear in the quantum fields, and therefore they are present only at the classical level, which means that they do not get quantum corrections.

The equations (4.1) and (4.3) can be solved by introducing the reduced action $\hat{\Sigma}$, defined by

$$\Sigma = \hat{\Sigma} + \int d^4x \left(b^a \partial A^a + \frac{\theta}{2} b^2 \right) + \xi^\mu \Delta_\mu , \quad (4.10)$$

where Δ_μ is given by (4.4).

It follows that

$$\frac{\delta \hat{\Sigma}}{\delta b^a} = \frac{\partial \hat{\Sigma}}{\partial \xi^\mu} = 0 , \quad (4.11)$$

and, for $\theta=0$,

$$\frac{\delta \hat{\Sigma}}{\delta \bar{c}^a} + \partial^\mu \frac{\delta \hat{\Sigma}}{\delta \Omega^{a\mu}} = 0 , \quad (4.12)$$

namely $\hat{\Sigma}$ does not depend on the Lagrange multiplier b nor on the global ghost ξ^μ , and, in the Landau gauge, it depends on the ghost \bar{c} and on the external source Ω^μ only through the combination

$$\eta^{\alpha\mu} \equiv \partial^\mu \bar{c}^a + \Omega^{a\mu} . \quad (4.13)$$

The reduced action $\hat{\Sigma}$ is recognized to be

$$\hat{\Sigma} = S_{SYM} + S_{matter} + S_{int}$$

$$\begin{aligned}
& + \int d^4x \left(M^a(f^{abc}c^bA^c + \bar{\varepsilon}^i\lambda_i^a) + N^a(f^{abc}c^bB^c + i\bar{\varepsilon}^i\gamma_5\lambda_i^a) + \eta^{a\mu}(-(D_\mu c)^a + \bar{\varepsilon}^i\gamma_\mu\lambda_i^a) \right. \\
& + \bar{\Lambda}^{a\alpha i}(f^{abc}c^b\lambda_{\alpha i}^c + \frac{1}{2}F_{\mu\nu}^a(\sigma^{\mu\nu}\varepsilon_i)_\alpha - (D_\mu A)^a(\gamma^\mu\varepsilon_i)_\alpha + i(D_\mu B)^a(\gamma^\mu\gamma_5\varepsilon_i)_\alpha + if^{abc}A^bB^c(\gamma_5\varepsilon_i)_\alpha) \\
& + L^a(\frac{1}{2}f^{abc}c^b c^c - (\bar{\varepsilon}^i\gamma^\mu\varepsilon_i)A_\mu^a - i(\bar{\varepsilon}^i\gamma_5\varepsilon_i)B^a + (\bar{\varepsilon}^i\varepsilon_i)A^a) \\
& + U_{iA}^*((T^a)_B^A c^a A^{iB} + \bar{\varepsilon}^i\psi^A) + U^{iA}(-(T^a)_A^B c^a A_{iB}^* + \bar{\psi}_A\varepsilon_i) \\
& + \bar{\Psi}_A^\alpha((T^a)_B^A c^a \psi_\alpha^B - 2(D_\mu A^i)^A(\gamma^\mu\varepsilon_i)_\alpha + 2(T^a)_B^A A^{iB}A^a\varepsilon_{\alpha i} - 2i(T^a)_B^A A^{iB}B^a(\gamma_5\varepsilon_i)_\alpha) \\
& + (-(T^a)_A^B c^a \bar{\psi}_B^\alpha + 2(D_\mu A_i^*)_A(\bar{\varepsilon}^i\gamma^\mu)^\alpha + 2(T^a)_A^B A_{iB}^*A^a\bar{\varepsilon}^{\alpha i} - 2i(T^a)_A^B A_{iB}^*B^a(\bar{\varepsilon}^i\gamma_5)^\alpha)\Psi_\alpha^A \\
& - (\bar{\varepsilon}^j\Lambda_i^a)(\bar{\Lambda}^{ai}\varepsilon_j) + \frac{1}{2}(\bar{\varepsilon}^j\Lambda_j^a)(\bar{\Lambda}^{ai}\varepsilon_i) \\
& \left. - (\bar{\varepsilon}^i\varepsilon_i)(\bar{\Psi}_A\Psi^A) + (\bar{\varepsilon}^i\gamma_5\varepsilon_i)(\bar{\Psi}_A\gamma_5\Psi^A) + (\bar{\varepsilon}^i\gamma^\mu\varepsilon_i)(\bar{\Psi}_A\gamma_\mu\Psi^A) \right) .
\end{aligned} \tag{4.14}$$

Because of the constraints (4.11) and (4.12), the functional $\hat{\Sigma}$ satisfies the modified Slavnov identity

$$\hat{\mathcal{S}}(\hat{\Sigma}) = 0 , \tag{4.15}$$

where

$$\begin{aligned}
\hat{\mathcal{S}}(\hat{\Sigma}) = \int d^4x \left(\frac{\delta\hat{\Sigma}}{\delta\eta^{a\mu}}\frac{\delta\hat{\Sigma}}{\delta A_\mu^a} + \frac{\delta\hat{\Sigma}}{\delta L^a}\frac{\delta\hat{\Sigma}}{\delta c^a} + \frac{\delta\hat{\Sigma}}{\delta M^a}\frac{\delta\hat{\Sigma}}{\delta A^a} + \frac{\delta\hat{\Sigma}}{\delta N^a}\frac{\delta\hat{\Sigma}}{\delta B^a} + \frac{\delta\hat{\Sigma}}{\delta\bar{\Lambda}^{ai}}\frac{\delta\hat{\Sigma}}{\delta\lambda_i^a} \right. \\
\left. + \frac{\delta\hat{\Sigma}}{\delta U_{iA}^*}\frac{\delta\hat{\Sigma}}{\delta A^{iA}} + \frac{\delta\hat{\Sigma}}{\delta U^{iA}}\frac{\delta\hat{\Sigma}}{\delta A_{iA}^*} + \frac{\delta\hat{\Sigma}}{\delta\bar{\Psi}_A}\frac{\delta\hat{\Sigma}}{\delta\psi^A} + \frac{\delta\hat{\Sigma}}{\delta\Psi^A}\frac{\delta\hat{\Sigma}}{\delta\bar{\psi}_A} \right) , \tag{4.16}
\end{aligned}$$

whose linearized Slavnov operator

$$\begin{aligned}
\hat{\mathcal{B}}_\Sigma = \int d^4x \left(\frac{\delta\hat{\Sigma}}{\delta\eta^{a\mu}}\frac{\delta}{\delta A_\mu^a} + \frac{\delta\hat{\Sigma}}{\delta A_\mu^a}\frac{\delta}{\delta\Omega^{a\mu}} + \frac{\delta\hat{\Sigma}}{\delta L^a}\frac{\delta}{\delta c^a} + \frac{\delta\hat{\Sigma}}{\delta c^a}\frac{\delta}{\delta L^a} + \frac{\delta\hat{\Sigma}}{\delta M^a}\frac{\delta}{\delta A^a} + \frac{\delta\hat{\Sigma}}{\delta A^a}\frac{\delta}{\delta M^a} \right. \\
\left. + \frac{\delta\hat{\Sigma}}{\delta N^a}\frac{\delta}{\delta B^a} + \frac{\delta\hat{\Sigma}}{\delta B^a}\frac{\delta}{\delta N^a} + \frac{\delta\hat{\Sigma}}{\delta\bar{\Lambda}^{ai}}\frac{\delta}{\delta\lambda_i^a} - \frac{\delta\hat{\Sigma}}{\delta\lambda_i^a}\frac{\delta}{\delta\bar{\Lambda}^{ai}} + \frac{\delta\hat{\Sigma}}{\delta U_{iA}^*}\frac{\delta}{\delta A^{iA}} + \frac{\delta\hat{\Sigma}}{\delta A^{iA}}\frac{\delta}{\delta U_{iA}^*} \right) \tag{4.17}
\end{aligned}$$

$$+\frac{\delta\hat{\Sigma}}{\delta U^{iA}}\frac{\delta}{\delta A_{iA}^*}+\frac{\delta\hat{\Sigma}}{\delta A_{iA}^*}\frac{\delta}{\delta U^{iA}}+\frac{\delta\hat{\Sigma}}{\delta\bar{\Psi}_A}\frac{\delta}{\delta\psi^A}-\frac{\delta\hat{\Sigma}}{\delta\psi^A}\frac{\delta}{\delta\bar{\Psi}_A}+\frac{\delta\hat{\Sigma}}{\delta\Psi^A}\frac{\delta}{\delta\bar{\psi}_A}-\frac{\delta\hat{\Sigma}}{\delta\bar{\psi}_A}\frac{\delta}{\delta\Psi^A}\Big)$$

is such that

$$\hat{\mathcal{B}}_\Sigma\hat{\Sigma}=0 \tag{4.18}$$

and therefore it is nilpotent

$$\hat{\mathcal{B}}_\Sigma\hat{\mathcal{B}}_\Sigma=0 \ . \tag{4.19}$$

Finally, in this framework the ghost equation of the Landau gauge becomes

$$\mathcal{F}^a\longrightarrow\hat{\mathcal{F}}^a=\int d^4x\frac{\delta}{\delta c^a} \ . \tag{4.20}$$

5 Conclusions

In this paper we gave an off-shell formulation of $N = 2$ Super Yang – Mills theories, without using auxiliary fields. The main result we obtained is represented by the Slavnov identity (3.14)

$$\mathcal{S}(\Sigma)=0 \ , \tag{5.1}$$

which is a generalized version of the ordinary one corresponding to the BRS symmetry. It describes the invariance of the theory not only under the BRS transformations (3.1), but also under the supersymmetry (2.2)-(2.3) and the translations (4.5). From (5.1) follows

the off-shell nilpotency of the linearized Slavnov operator

$$\mathcal{B}_\Sigma \mathcal{B}_\Sigma = 0 \ , \tag{5.2}$$

which contains all the informations concerning the algebraic structure of the theory, namely the nilpotency of the BRS operator and the *off-shell* closure of the supersymmetry commutators on the translations. This result has been obtained without making use of auxiliary fields and therefore this method is quite general, being applicable also when no off-shell formulation in terms of auxiliary fields exists, like the $N = 4$ case. The point is that the problem of finding auxiliary fields is circumvented in this formulation, because in order to study the quantum extension of a theory characterized by nonlinear symmetries, it is necessary to couple external sources to the nonlinear variations of the quantum fields. These external sources transform into the equations of motion of the corresponding quantum fields, therefore in this framework they do precisely the same job as the conventional auxiliary fields. Even when a complete set of auxiliary fields can be introduced, as it happens for the $N = 1$ and $N = 2$ cases, the alternative formulation described in this paper avoids the inconvenients arising from the infinite chain of external sources introduced to put the formalism off-shell, thus making easy the otherwise impossible quantization of the model.

Moreover, for what concerns the renormalization of the theory, the lack of an acceptable regularization preserving both BRS and supersymmetry renders the algebraic method of renormalization even more necessary than usual [3]. Formally, the $N = 2$

SYM theory is now completely described by a Slavnov identity and the constraints discussed in Section 4. The algebraic renormalization of such a model, although technically rather involved, can now be carried on as if we were dealing with an ordinary gauge field theory [9].

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